

OPERATOR-VALUED FREE FISHER INFORMATION OF RANDOM MATRICES

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ABSTRACT. We study the operator-valued free Fisher information of random matrices in an operator-valued noncommutative probability space. We obtain a formula for $\Phi_{M_2(\mathcal{B})}^*(A, A^*, M_2(\mathcal{B}), \eta)$, where $A \in M_2(\mathcal{B})$ is a 2×2 operator matrix on \mathcal{B} , and η is linear operators on $M_2(\mathcal{B})$. Then we consider a special setting: A is an operator-valued semicircular matrix with conditional expectation covariance, and find that $\Phi_{\mathcal{B}}^*(c, c^* : \mathcal{B}, id) = 2Index(E)$, where E is a conditional expectation of \mathcal{B} onto \mathcal{D} and c is a circular variable with covariance E .

1. INTRODUCTION AND PRELIMINARIES

Free probability theory is a noncommutative probability theory where the classical concept of independence is replaced by the notion of "freeness". This theory, due to D.Voiculescu, has very important applications on operator algebras (see [6, 21])

Originally, a noncommutative probability space is a pair (\mathcal{A}, τ) , where \mathcal{A} is a C^* - or von Neumann algebra and τ is a state on \mathcal{A} . Free independence is defined in terms of reduced free product relation given by τ . This notion was generalized by D.Voiculescu and others to an algebra valued noncommutative probability space where τ is replaced by a conditional expectation $E_{\mathcal{B}}$ onto a subalgebra \mathcal{B} of \mathcal{A} , and freeness is replaced by freeness with amalgamation.

Definition 1. [20] Let \mathcal{A} be a unital algebra over \mathbb{C} , and let \mathcal{B} be a subalgebra of \mathcal{A} , $1 \in \mathcal{B}$. $E_{\mathcal{B}} : \mathcal{A} \rightarrow \mathcal{B}$ is a conditional expectation, i.e. a linear map such that $E_{\mathcal{B}}(b_1 a b_2) = b_1 E_{\mathcal{B}}(a) b_2$, $E_{\mathcal{B}}(b) = b$, for any $b, b_1, b_2 \in \mathcal{B}$, $a \in \mathcal{A}$. We call $(\mathcal{A}, E_{\mathcal{B}}, \mathcal{B})$ an operator-valued (or \mathcal{B} -valued) noncommutative probability space and elements in \mathcal{A} are called \mathcal{B} - random variables.

The algebra freely generated by \mathcal{B} and an indeterminate X will be denoted by $\mathcal{B}[X]$. The distribution of $a \in \mathcal{A}$ is a conditional expectation $\mu_a : \mathcal{B}[X] \rightarrow \mathcal{B}$, $\mu_a = E_{\mathcal{B}} \circ \tau_a$, where $\tau_a : \mathcal{B}[X] \rightarrow \mathcal{A}$ is the unique homomorphism such that $\tau_a(b) = b$, for any $b \in \mathcal{B}$, $\tau_a(X) = a$. Let $\mathcal{B} \subset \mathcal{A}_i \subset \mathcal{A}$, $(i \in I)$ be subalgebras. The family $(\mathcal{A}_i)_{i \in I}$ will be called free with amalgamation over \mathcal{B} (or \mathcal{B} -free), if $E_{\mathcal{B}}(a_1 a_2 \cdots a_n) = 0$ whenever $a_j \in \mathcal{A}_{i_j}$ with $i_1 \neq i_2 \neq \cdots \neq i_n$ and $E_{\mathcal{B}}(a_j) = 0$, $1 \leq j \leq n$. A sequence $\{a_i\}_{i \in I} \subseteq \mathcal{A}$ will be called free with amalgamation over \mathcal{B} if the family of subalgebras generated by $(\mathcal{B} \cup \{a_i\})_{i \in I}$ is \mathcal{B} -free.

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In this paper we only consider \mathcal{B} -valued W^* -noncommutative probability space, which means \mathcal{A}, \mathcal{B} are von Neumann (sub)algebras and $E_{\mathcal{B}}$ is a faithful conditional expectation (i.e. projection with norm 1) of \mathcal{A} onto \mathcal{B} .

Classical Fisher information is derived from the statistical estimation theory which is defined by R.A.Fisher (for some details we refer to [2]). By analogy with the classical case D.Voiculescu introduced the free Fisher information of self-adjoint random variables in a tracial W^* -noncommutative probability space (see [22, 23]). Then A.Nica, D.Shlykhtenko and R.Speicher investigated it in view of cumulant function (see [11]) and study the free Fisher information of random matrices (see [12]). They found the following equality holds: $\Phi^*(\{a_{ij}, a_{ij}^*\}_{1 \leq i, j \leq d} : \mathcal{B}) = d^3 \Phi^*(\{A, A^*\} : M_d(\mathcal{B}))$, where Φ^* denotes the free Fisher information, $A = [a_{ij}]_{i,j=1}^d \in M_d(\mathcal{A}) := \mathcal{A} \otimes M_d(\mathbb{C})$ is an operator matrix with entries in \mathcal{A} , and $\mathcal{B} \subseteq \mathcal{A}$ is a subalgebra of \mathcal{A} . D.Shlykhtenko even introduced free Fisher information with respect to a completely positive map (see [15]). All of these works are done in the tracial von Neumann algebra framework. In the case of states, the non-tracial framework was worked out by D. Shlyakhtenko in [17]. In [9, 10], we generalized the notion of free Fisher information to the operator-valued setting and this work is done in the general von Neumann algebra framework and found that the free Fisher information of an operator-valued semicircular variable with conditional expectation covariance is closely related to modular frames.

In the present paper, based on the notion of operator-valued free Fisher information with respect to a map which introduced in [10], we study the free Fisher information of random matrices and obtain a formula of operator-valued free Fisher information of random matrices, which generalized Theorem 1.2, Proposition 4.1 in [12], and the method we prove it is quite different from that of Theorem 1.2 in [12], since the conditional expectations are not "tracial" in general. Furthermore, we find that the free Fisher information of circular variables is closely related to modular frames and the index of conditional expectations.

Our main method to prove the results is R.Speicher's cumulant function technique (see [14]), just as in [9, 10]. We use $\widehat{k}_{\mathcal{B}} := (k_{\mathcal{B}}^{(n)})_{n \geq 1}$ and $\widehat{E}_{\mathcal{B}} := (E_{\mathcal{B}}^{(n)})$ to denote the cumulant and moment functions induced by $E_{\mathcal{B}}$ respectively where $E_{\mathcal{B}}^{(n)}(a_1 \otimes \cdots \otimes a_n) := E_{\mathcal{B}}(a_1 \cdots a_n)$ and $k_{\mathcal{B}}^{(n)}$ is determined by the following recurrence formula (where $E_{\mathcal{B}}^{(0)} = 1$ formally):

$$\begin{aligned} E_{\mathcal{B}}^{(n)}(a_1 \otimes \cdots \otimes a_n) &= \\ &= \sum_{r=0}^{n-1} \sum_{1 < i(1) < i(2) \cdots < i(r) \leq n} k_{\mathcal{B}}^{(r+1)}(a_1 E_{\mathcal{B}}^{(i(1)-2)}(a_1 \otimes \\ &\quad \cdots \otimes a_{i(1)-1}) \otimes a_{i(1)} E_{\mathcal{B}}^{(i(2)-i(1)-1)}(a_{i(1)+1} \otimes \cdots \otimes a_{i(2)-1}) \\ &\quad \otimes a_{i(2)} \cdots \otimes a_{i(r)} E_{\mathcal{B}}^{(n-i(r))}(a_{i(r)+1} \otimes \cdots \otimes a_n)) \end{aligned}$$

for all $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathcal{A}$. It is easy to see $\widehat{E}_{\mathcal{B}}$ and $\widehat{k}_{\mathcal{B}}$ determine each other uniquely by the above recurrence formula.

Another tools we use is the frame theory. The frame theory in Hilbert space is derived from the wavelets theory. D.Larson and D.Han etc. studied frames by the operator theory (see [7]). Furthermore, M.Frank and D.Larson generalized the frame notion to the Hilbert C^* -module setting called modular frames (see [4, 5]). For an unital C^* -algebra \mathcal{D} , a Hilbert \mathcal{D} -module is a linear space and algebraic

\mathcal{D} -module \mathcal{M} together with a \mathcal{D} -valued inner product $\langle \cdot, \cdot \rangle_{\mathcal{D}}$ and complete with respect to the Hilbert C^* -module norm: $\|\langle \cdot, \cdot \rangle_{\mathcal{D}}\|^{\frac{1}{2}}$. We refer to [8] for more details on Hilbert C^* -module theory. A sequence $\{f_j : j \in \mathbb{J}\} \subseteq \mathcal{M}$ is said to be a frame if there are real constants $C, D > 0$ such that

$$C\langle x, x \rangle_{\mathcal{D}} \leq \sum_i \langle x, f_i \rangle_{\mathcal{D}} \langle f_i, x \rangle_{\mathcal{D}} \leq D\langle x, x \rangle_{\mathcal{D}}, \text{ for all } x \in \mathcal{M}.$$

If one can choose $C = D = 1$ the frame will be called normalized tight. Note that for a frame the sum in the middle only converges weakly, in general. If the sum converges in norm then the frame will be called standard (see [4]). M. Frank and D. Larson also proved that every algebraically finitely or countably generated Hilbert C^* -module possesses a standard normalized tight frame ([4]). In this paper, when we say "frames", it means "standard frames".

The following lemma will be used frequently.

Lemma 2. [9] *Let $(\mathcal{A}, E_{\mathcal{B}}, \mathcal{B})$ be a noncommutative probability space, $E : \mathcal{B} \rightarrow \mathcal{D}$ be a conditional expectation, and let $X \in (\mathcal{A}, E_{\mathcal{B}}, \mathcal{B})$ be a semicircular variable with covariance E . Denote the Hilbert C^* -module generated by \mathcal{B} by $\mathcal{L}_{\mathcal{D}}^2(\mathcal{B})$ whose inner product induced by E and let $\mathcal{L}_{\mathcal{D}}^*(\mathcal{B}) = \mathcal{B}$. Let $\{f_i\} \subseteq \mathcal{B}$ be a frame in $\mathcal{L}_{\mathcal{D}}^2(\mathcal{B})$. Then*

$$(1) \quad \Phi_{\mathcal{B}}^*(X : \mathcal{B}) = \sum_i f_i f_i^* = \text{Index}(E)$$

and X is free from \mathcal{A} with amalgamation over \mathcal{D} .

In the above lemma, $\text{Index}(E)$ denotes the index of the conditional expectation E . We refer to [1, 3, 18] for more details on the index of conditional expectations.

The paper is organized as follows: In section 2, we review the definition and some results of the operator-valued free Fisher information with respect to a linear map. We get a explicit formula for the operator-valued free Fisher information of random matrices with respect to a certain kind of linear maps. In section 3, we consider the semicircular random matrices' free Fisher information. We can compute the free Fisher information of circular variables with covariance E by modular frames, and furthermore, we point out that it is just equal to the double of index of E .

2. THE FREE FISHER INFORMATION OF RANDOM MATRICES

In [10], we introduced the operator-valued free Fisher information of one variable with respect to a linear map which is a generalization of D.Shlyakhtenko's notion in [15]. Such a notion can be generalized to several random variables'setting easily.

Definition 3. *Let $(\mathcal{A}, E_{\mathcal{B}}, \mathcal{B})$ be a \mathcal{B} -valued W^* -noncommutative probability space. Suppose that $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{A}$ is a von Neumann subalgebra and $\{X_i\}_{i=1}^n \subseteq \mathcal{A}$ is algebraically free from \mathcal{C} over \mathcal{B} . Denote by $\mathcal{L}_{\mathcal{B}}^2(\mathcal{C}[X_1, X_2, \dots, X_n])$ the Hilbert \mathcal{B} -module generated by $\mathcal{C}[X_1, X_2, \dots, X_n]$, whose inner product is defined by $\langle x, y \rangle := E_{\mathcal{B}}(x^*y)$, for any $x, y \in \mathcal{C}[X_1, X_2, \dots, X_n]$. Let $\{\eta_i\}_{i=1}^n$ be a sequence of linear maps on \mathcal{C} . $\{\xi_i\}_{i=1}^n \subseteq \mathcal{L}_{\mathcal{B}}^2(\mathcal{C}[X_1, X_2, \dots, X_n])$ will be called the conjugate system of $\{X_i\}_{i=1}^n$ with respect to \mathcal{C} , $\{\eta_i\}_{i=1}^n$, if it satisfying:*

$$(2) \quad \begin{aligned} & E_{\mathcal{B}}(\xi_i c_0 X_{i_1} c_1 X_{i_2} \cdots X_{i_m} c_m) \\ &= \sum_{j=1}^m \delta_{i, i_j} E_{\mathcal{B}}(\eta_i E_{\mathcal{C}}(c_0 X_{i_1} \cdots X_{i_{j-1}} c_{j-1})) \cdot E_{\mathcal{B}}(c_j X_{i_{j+1}} \cdots X_{i_m} c_m) \end{aligned}$$

for any $m \geq 0$, $c_0, \dots, c_m \in \mathcal{C}$ and $i_1, \dots, i_m \in \{1, 2, \dots, m\}$.

If such a $\{\xi_i\}_{i=1}^n$ exists, then we define $\Phi_{\mathcal{B}}^*(X_1, \dots, X_n : \mathcal{C}, \eta_1, \dots, \eta_n)$ —the free Fisher information of $\{X_i\}_{i=1}^n$ with respect to $\mathcal{C}, \{\eta_i\}_{i=1}^n$ by $\sum_i \xi_i \xi_i^*$, that is,

$$\Phi_{\mathcal{B}}^*(X_1, \dots, X_n : \mathcal{C}, \eta_1, \dots, \eta_n) = \sum_{i=1}^n \xi_i \xi_i^*$$

From [10, 11], we know that the conjugate system can be expressed by cumulant function.

Proposition 4. *With the above notations, let $E_{\mathcal{C}}$ be a conditional expectation of \mathcal{A} onto \mathcal{C} such that $E_{\mathcal{B}} = E_{\mathcal{B}} E_{\mathcal{C}}$ and let $(k_{\mathcal{C}}^{(n)})_{n \geq 1}$ be the cumulant function induced by $E_{\mathcal{C}}$. Then $\{\xi_i\}_{i=1}^n \subseteq \mathcal{L}_{\mathcal{B}}^2(\mathcal{C}[X_1, X_2, \dots, X_n])$ is the conjugate system of $\{X_i\}_{i=1}^n$ with respect to $\mathcal{C}, \{\eta_i\}_{i=1}^n$ if and only if the following equations hold*

$$\begin{cases} k_{\mathcal{C}}^{(1)}(\xi_i c) = 0, & \forall i \in \{1, 2, \dots, n\}, \\ k_{\mathcal{C}}^{(2)}(\xi_i \otimes ca) = \delta_{a, X_i} E_{\mathcal{B}}(\eta_i(c)), & \forall i \in \{1, 2, \dots, n\}, \\ k_{\mathcal{C}}^{(m+1)}(\xi_i \otimes c_1 a_1 \otimes \dots \otimes c_m a_m) = 0, & \forall i \in \{1, 2, \dots, n\}, m \geq 2 \end{cases}$$

where $c, c_1, \dots, c_m \in \mathcal{C}$, $a, a_1, \dots, a_m \in \{X_1, \dots, X_n\} \cup \mathcal{C}$.

We introduce some notations about random matrices. $M_d(\mathcal{A}) := \mathcal{A} \otimes M_d(\mathbb{C})$, $M_d(\mathcal{B}) := \mathcal{B} \otimes M_d(\mathbb{C})$ are the sets of all the $d \times d$ matrices on \mathcal{A}, \mathcal{B} respectively. If $E_{\mathcal{B}}$ is a conditional expectation of \mathcal{A} onto \mathcal{B} , then define the conditional expectation $E_{\mathcal{B}} \otimes I_d : \mathcal{A} \otimes M_d(\mathbb{C}) \rightarrow \mathcal{B} \otimes M_d(\mathbb{C})$ by $(E_{\mathcal{B}} \otimes I_d)([a_{ij}]_{i,j=1}^d) := [E_{\mathcal{B}}(a_{ij})]_{i,j=1}^d$, for all $A := [a_{ij}]_{i,j=1}^d \in M_d(\mathcal{A})$. Obviously, $(M_d(\mathcal{A}), E_{\mathcal{B}} \otimes I_d, M_d(\mathcal{B}))$ is a $M_d(\mathcal{B})$ -valued noncommutative probability and the elements in it are called operator-valued random matrices in general.

Since there is no essential difference between 2×2 and $n \times n$ matrices, for convenience, we only consider 2×2 matrices below.

The following theorem describe the free Fisher information of random matrices in terms of its entries' free Fisher information.

Theorem 5. *Let $(\mathcal{A}, E_{\mathcal{B}}, \mathcal{B})$ be a \mathcal{B} -valued W^* -noncommutative probability space and let $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{A}$ be a subalgebra. $\{\eta_{ij}, \xi_{ij}\}_{i,j=1}^2$ is a sequence of linear map on \mathcal{C} .*

Define $\eta : M_2(\mathcal{C}) \rightarrow M_2(\mathcal{C})$ by $\eta \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} \eta_{11}(c_{11}) + \eta_{21}(c_{22}) & 0 \\ 0 & \eta_{12}(c_{11}) + \eta_{22}(c_{22}) \end{pmatrix}$

and $\xi : M_2(\mathcal{C}) \rightarrow M_2(\mathcal{C})$ by $\xi \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} \xi_{11}(c_{11}) + \xi_{12}(c_{22}) & 0 \\ 0 & \xi_{21}(c_{11}) + \xi_{22}(c_{22}) \end{pmatrix}$.

$A := [a_{ij}]_{i,j=1}^2 \in M_2(\mathcal{A})$. Let $\{x_{ij}, y_{ij}\}_{i,j=1}^2$ be the conjugate system of $\{a_{ij}, a_{ij}^\}_{i,j=1}^2$ with respect to $\{\eta_{ij}, \xi_{ij}\}_{i,j=1}^2$. Then*

$$\Phi_{M_2(\mathcal{B})}^*(A, A^* : M_2(\mathcal{C}), \eta, \xi) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where

$$\begin{aligned} A_{11} &= \Phi_{\mathcal{B}}^*(a_{11}, a_{11}^* : \mathcal{C}, \eta_{11}, \xi_{11}) + E_{\mathcal{B}}(x_{21} x_{21}^*) + E_{\mathcal{B}}(y_{12} y_{12}^*) \\ A_{12} &= E_{\mathcal{B}}(x_{11} x_{12}^* + x_{21} x_{22}^* + y_{11} y_{21}^* + y_{12} y_{22}^*) \\ A_{21} &= E_{\mathcal{B}}(x_{12} x_{11}^* + x_{22} x_{21}^* + y_{21} y_{11}^* + y_{22} y_{12}^*) \\ A_{22} &= \Phi_{\mathcal{B}}^*(a_{22}, a_{22}^* : \mathcal{C}, \eta_{22}, \xi_{22}) + E_{\mathcal{B}}(x_{12} x_{12}^*) + E_{\mathcal{B}}(y_{21} y_{21}^*) \end{aligned}$$

Proof. Let $X_1 := \begin{pmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{pmatrix}$, $X_2 := \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$. Below we show $\{X_1, X_2\}$ is the conjugate system of $\{A, A^*\}$ with respect to $\{\eta, \xi\}$ using Proposition 4.

Denote by $(K^{(n)})_{n \geq 1}$ the cumulant function induced by $(E_C \otimes I_2)$. Then for any $C_k := [c_{ij}^{(k)}]_{i,j=1}^2$, we get the following equalities.

$$\begin{aligned} K^{(1)}(X_1 C_1) &= (E_C \otimes I_2) \begin{pmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{pmatrix} \begin{pmatrix} c_{11}^{(1)} & c_{12}^{(1)} \\ c_{21}^{(1)} & c_{22}^{(1)} \end{pmatrix} \\ &= (E_C \otimes I_2) \begin{pmatrix} x_{11}c_{11}^{(1)} + x_{21}c_{21}^{(1)} & x_{11}c_{12}^{(1)} + x_{21}c_{22}^{(1)} \\ x_{12}c_{11}^{(1)} + x_{22}c_{21}^{(1)} & x_{12}c_{12}^{(1)} + x_{22}c_{22}^{(1)} \end{pmatrix} \\ &= 0; \end{aligned}$$

By the same way, we can prove $K^{(1)}(X_2 C_1) = 0$.

$$\begin{aligned} K^{(2)}(X_1 \otimes C_1 A) &= (E_C \otimes I_2)^{(2)}(X_1 \otimes C_1 A) - (E_C \otimes I_2)^{(1)}(X_1 (E_C \otimes I_2)(C_1 A)) \\ &= (E_C \otimes I_2)(X_1 C_1 A) \\ &= (E_C \otimes I_2) \begin{pmatrix} \sum_{i_1, i_2=1}^2 x_{i_1 1} c_{i_1 i_2}^{(1)} a_{i_2 1} & \sum_{i_1, i_2=1}^2 x_{i_1 1} c_{i_1 i_2}^{(1)} a_{i_2 2} \\ \sum_{i_1, i_2=1}^2 x_{i_1 2} c_{i_1 i_2}^{(1)} a_{i_2 1} & \sum_{i_1, i_2=1}^2 x_{i_1 2} c_{i_1 i_2}^{(1)} a_{i_2 2} \end{pmatrix} \\ &= \begin{pmatrix} E_B(\eta_{11}(c_{11}^{(1)}) + \eta_{21}(c_{22}^{(1)})) & 0 \\ 0 & E_B(\eta_{12}(c_{11}^{(1)}) + \eta_{22}(c_{22}^{(1)})) \end{pmatrix} \\ &= (E_B \otimes I_2)(\eta(C_1)) \end{aligned}$$

Similarly, we get $K^{(2)}(X_2 \otimes C_1 A^*) = E_B(\xi(C_1))$; $K^{(2)}(X_1 \otimes C_1 A^*) = 0$; $K^{(2)}(X_2 \otimes C_1 A) = 0$.

To prove

$$(3) \quad K^{(m+1)}(X_1 \otimes C_1 A_1 \otimes \cdots \otimes C_m A_m) = 0, \forall m \geq 2,$$

where $A_i \in \{A, A^*, M_2(\mathcal{C})\}$ we need to induce on m .

When $m = 2$,

$$\begin{aligned} K^{(3)}(X_1 \otimes C_1 A_1 \otimes C_2 A_2) &= (E_C \otimes I_2)(X_1 \cdot C_1 A_1 \cdot C_2 A_2) - K^{(2)}(X_1 \otimes C_1 A_1 (E_C \otimes I_2)(C_2 A_2)) \\ &\quad - K^{(2)}(X_1 \otimes (E_C \otimes I_2)(C_1 A_1) C_2 A_2) \\ &= (E_C \otimes I_2)(X_1 \cdot C_1 A_1 \cdot C_2 A_2) - \delta_{A_1, A}(E_C \otimes I_2)(\eta(C_1))(E_C \otimes I_2)(C_2 A_2) \\ &\quad - \delta_{A_2, A}(E_C \otimes I_2)\eta((E_C \otimes I_2)(C_1 A_1) C_2) \end{aligned}$$

On the other hand,

$$\begin{aligned}
& (E_C \otimes I_2)(X_1 \cdot C_1 A_1 \cdot C_2 A_2) \\
&= (E_C \otimes I_2) \left[\begin{pmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{pmatrix} \begin{pmatrix} c_{11}^{(1)} & c_{12}^{(1)} \\ c_{21}^{(1)} & c_{22}^{(1)} \end{pmatrix} \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} \end{pmatrix} \begin{pmatrix} c_{11}^{(2)} & c_{12}^{(2)} \\ c_{21}^{(2)} & c_{22}^{(2)} \end{pmatrix} \begin{pmatrix} a_{11}^{(2)} & a_{12}^{(2)} \\ a_{21}^{(2)} & a_{22}^{(2)} \end{pmatrix} \right] \\
&= \left[\sum_{i_3, i_4, i_5, i_6=1}^2 E_C^{(3)}(x_{i_3 i_1} \otimes (c_{i_3 i_4}^{(1)} a_{i_4 i_5}^{(1)}) \otimes (c_{i_5 i_6}^{(2)} a_{i_6 i_2}^{(2)})) \right]_{i_1, i_2=1}^2 \\
&= \left[\sum_{i_3, i_4, i_5, i_6=1}^2 k^{(1)}(x_{i_3 i_1} E_C(c_{i_3 i_4}^{(1)} a_{i_4 i_5}^{(1)} c_{i_5 i_6}^{(2)} a_{i_6 i_2}^{(2)})) \right. \\
&\quad + k^{(2)}(x_{i_3 i_1} \otimes c_{i_3 i_4}^{(1)} a_{i_4 i_5}^{(1)} E_C(c_{i_5 i_6}^{(2)} a_{i_6 i_2}^{(2)})) \\
&\quad \left. + k^{(3)}(x_{i_3 i_1} E_C(c_{i_3 i_4}^{(1)} a_{i_4 i_5}^{(1)}) \otimes c_{i_5 i_6}^{(2)} a_{i_6 i_2}^{(2)}) + k^{(3)}(x_{i_3 i_1} \otimes c_{i_2 i_3}^{(1)} a_{i_3 i_4}^{(1)} \otimes c_{i_4 i_5}^{(2)} a_{i_5 i_1}^{(2)}) \right]_{i_1, i_2=1}^2 \\
&= \delta_{A_1 A}(E_B \otimes I_2)(\eta(C_1))(E_C \otimes I_2)(C_2 A_2) + \delta_{A_2 A}(E_B \otimes I_2)\eta((E_C \otimes I_2)(C_1 A_1)C_2),
\end{aligned}$$

$$\text{so, } K^{(3)}(X_1 \otimes C_1 A_1 \otimes C_2 A_2) = 0$$

Now assume that the Eq.(3) holds for the numbers $\leq m-1$, then

$$\begin{aligned}
 & K^{(m+1)}(X_1 \otimes C_1 A_1 \otimes \cdots \otimes C_m A_m) \\
 &= (E_C \otimes I_2)(X_1 \cdot C_1 A_1 \cdots C_m A_m) \\
 &\quad - \sum_{j=1}^{m+1} K^{(2)}(X_1 (E_C \otimes I_2)^{(j-1)} (C_1 A_1 \otimes \cdots \otimes C_{j-1} A_{j-1}) \\
 &\quad \otimes C_j A_j (E_C \otimes I_2)^{(m-j)} (C_{j+1} A_{j+1} \otimes \cdots \otimes C_m A_m)) \\
 &= (E_C \otimes I_2)(X_1 \cdot C_1 A_1 \cdots C_m A_m) \\
 &\quad - \sum_{j=1}^{m+1} \delta_{AA_j} (E_B \otimes I_2) (\eta(E_C \otimes I_2) (C_1 A_1 \cdots C_{j-1} A_{j-1}) C_j) \\
 &\quad (E_C \otimes I_2)^{(m-j)} (C_{j+1} A_{j+1} \otimes \cdots \otimes C_m A_m)) \\
 &= \left[\sum_{i_3, i_4, \dots, i_{2(m+1)}=1}^2 E_C(x_{i_3 i_1} (c_{i_3 i_4}^{(1)} a_{i_4 i_5}^{(1)}) \cdots (c_{i_{2(m+1)-1}}^{(m)} a_{i_{2(m+1)} i_2}^{(m)})) \right]_{i_1, i_2=1}^2 \\
 &\quad - \sum_{j=1}^{m+1} \delta_{AA_j} (E_B \otimes I_2) \eta((E_C \otimes I_2) (C_1 A_1 \cdots C_{j-1} A_{j-1}) C_j) \\
 &\quad (E_C \otimes I_2)^{(m-j)} (C_{j+1} A_{j+1} \otimes \cdots \otimes C_m A_m)) \\
 &= \left[\sum_{i_3, i_4, \dots, i_{2(m+1)}}^2 \sum_{j=0}^m k^{(2)}(x_{i_3 i_1} E_C(c_{i_3 i_4} a_{i_4 i_5}^{(1)} \cdots c_{i_{2(j+1)-1} i_{2(j+1)}}^{(j)} a_{i_{2(j+1)} i_{2(j+1)+1}}^{(j)})) \right. \\
 &\quad \left. \otimes c_{i_{2(j+2)-1} i_{2(j+2)}}^{(j+1)} a_{i_{2(j+2)} i_{2(j+2)}}^{(j+1)} E_C(c_{i_{2(j+2)+1} i_{2(j+2)+2}}^{(j+2)} a_{i_{2(j+2)+2} i_{2(j+2)+3}}^{(j+2)} \cdots c_{i_{2m+1} i_{2m+2}}^{(m)} a_{i_{2m+2} i_2}^{(m)})) \right]_{i_1, i_2=1}^2 \\
 &\quad - \sum_{j=1}^{m+1} \delta_{AA_j} (E_B \otimes I_2) \eta((E_C \otimes I_2) (C_1 A_1 \cdots C_{j-1} A_{j-1}) C_j) \\
 &\quad (E_C \otimes I_2)^{(m-j)} (C_{j+1} A_{j+1} \otimes \cdots \otimes C_m A_m)) \\
 &= \left[\sum_{i_3, \dots, i_{2(m+1)}=1}^2 \delta_{a_{i_3 i_1}, a_{i_{2(j+2)} i_{2(j+2)}}^{(j+1)}} E_B \eta(E_C(c_{i_3 i_4} a_{i_4 i_5}^{(1)} \cdots c_{i_{2(j+1)-1} i_{2(j+1)}}^{(j)} a_{i_{2(j+1)} i_{2(j+1)+1}}^{(j)})) \right. \\
 &\quad \left. c_{i_{2(j+2)-1} i_{2(j+2)}}^{(j+1)} E_C(c_{i_{2(j+2)+1} i_{2(j+2)+2}}^{(j+2)} a_{i_{2(j+2)+2} i_{2(j+2)+3}}^{(j+2)} \cdots c_{i_{2m+1} i_{2m+2}}^{(m)} a_{i_{2m+2} i_2}^{(m)})) \right]_{i_1, i_2=1}^2 \\
 &\quad - \sum_{j=1}^{m+1} \delta_{AA_j} (E_B \otimes I_2) \eta((E_C \otimes I_2) (C_1 A_1 \cdots C_{j-1} A_{j-1}) C_j) \\
 &\quad (E_C \otimes I_2)^{(m-j)} (C_{j+1} A_{j+1} \otimes \cdots \otimes C_m A_m)) \\
 &= 0
 \end{aligned}$$

Thus we claim that X_1, X_2 is the conjugate system of (A, A^*) with respect to (η, ξ) . So,

$$\begin{aligned} & \Phi_{\mathcal{B}}^*(A, A^* : M_2(\mathcal{C}), \eta, \xi) \\ &= (E_{\mathcal{B}} \otimes I_2) \left[\begin{pmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{pmatrix} \begin{pmatrix} x_{11}^* & x_{12}^* \\ x_{21}^* & x_{22}^* \end{pmatrix} + \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \begin{pmatrix} y_{11}^* & y_{21}^* \\ y_{12}^* & y_{22}^* \end{pmatrix} \right] \\ &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \end{aligned}$$

□

The following corollary is a analogue of Theorem 1.2 in [12].

Corollary 6. Let $\mathcal{B}_T^{(2)} := \left\{ \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \mid b \in \mathcal{B} \right\}$ and $(E_{\mathcal{B}} \otimes tr_T) \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} := \begin{pmatrix} E_{\mathcal{B}} \left(\frac{b_{11} + b_{22}}{2} \right) & 0 \\ 0 & E_{\mathcal{B}} \left(\frac{b_{11} + b_{22}}{2} \right) \end{pmatrix}$. Then in the $\mathcal{B}_T^{(2)}$ -valued noncommutative probability space $(M_2(\mathcal{A}), E_{\mathcal{B}} \otimes tr_T, \mathcal{B}_T^{(2)})$, for a random matrix $A = [a_{ij}]_{i,j=1}^2$ we have

$$\Phi_{\mathcal{B}_T^{(2)}}^*(A, A^* : M_2(\mathcal{C}), id) = \frac{1}{8} \begin{pmatrix} \sum_{i,j=1}^2 \Phi_{\mathcal{B}}^*(a_{ij}, a_{ij}^* : \mathcal{C}, id) & 0 \\ 0 & \sum_{i,j=1}^2 \Phi_{\mathcal{B}}^*(a_{ij}, a_{ij}^* : \mathcal{C}, id) \end{pmatrix}$$

Proof. With the notations as in Theorem and its proof, let $\eta_{ij} = \xi_{ij} = 1$, $i, j = 1, 2$. Then from the proof of Theorem, we know

$$\begin{aligned} K^{(2)}(\frac{1}{2}X_1 \otimes CA) &= \begin{pmatrix} \frac{E_{\mathcal{B}}(c_{11} + c_{22})}{2} & \\ & \frac{E_{\mathcal{B}}(c_{11} + c_{22})}{2} \end{pmatrix} \\ &= (E_{\mathcal{B}} \otimes tr_T)(C) \end{aligned}$$

and

$$\begin{aligned} K^{(2)}(\frac{1}{2}X_2 \otimes CA^*) &= \begin{pmatrix} \frac{E_{\mathcal{B}}(c_{11} + c_{22})}{2} & \\ & \frac{E_{\mathcal{B}}(c_{11} + c_{22})}{2} \end{pmatrix} \\ &= (E_{\mathcal{B}} \otimes tr_T)(C). \end{aligned}$$

Hence $\{\frac{1}{2}X_1, \frac{1}{2}X_2\}$ is the conjugate system of A, A^* with respect to $\mathcal{B}_T^{(2)}, id$ and

$$\begin{aligned} & \Phi_{\mathcal{B}_T^{(2)}}^*(A, A^* : M_2(\mathcal{C}), id) \\ &= \frac{1}{4}E_{\mathcal{B}} \otimes tr_T(X_1 X_1^*) + \frac{1}{4}E_{\mathcal{B}} \otimes tr_T(X_2 X_2^*) \\ &= \frac{1}{8} \begin{pmatrix} \sum_{i,j=1}^2 \Phi_{\mathcal{B}}^*(a_{ij}, a_{ij}^* : \mathcal{C}, id) & 0 \\ 0 & \sum_{i,j=1}^2 \Phi_{\mathcal{B}}^*(a_{ij}, a_{ij}^* : \mathcal{C}, id) \end{pmatrix} \end{aligned}$$

□

3. THE FREE FISHER INFORMATION OF RANDOM MATRICES WITH SEMICIRCULAR AND CIRCULAR ENTRIES

Recall that $X \in (\mathcal{A}, E_{\mathcal{B}}, \mathcal{B})$ will be called a semicircular variable with covariance η (or η -semicircular variable, where η is a linear map on \mathcal{B}), if it satisfies: $k_{\mathcal{B}}^{(1)}(X) = 0$, $k_{\mathcal{B}}^{(2)}(X \otimes bX) = \eta(b)$, $k_{\mathcal{B}}^{(m+1)}(X \otimes b_1 X \otimes \cdots \otimes b_m X) = 0$, $\forall m \geq 2$, where $(k_{\mathcal{B}}^{(n)})_{n \geq 1}$ is the cumulant function induced by $E_{\mathcal{B}}$. Let X, Y be two free η -semicircular variables, then $C := \frac{X + iY}{\sqrt{2}}$ will be called a circular variable in $(\mathcal{A}, E_{\mathcal{B}}, \mathcal{B})$ with covariance η (or η -circular).

In the classical free probability theory, it is well known that $\begin{pmatrix} f_1 & c \\ c^* & f_2 \end{pmatrix} \in (M_2(\mathcal{A}), \tau \otimes \text{tr}, \mathbb{C})$ is semicircular, where $f_1, f_2, \{c, c^*\}$ is a free family in (\mathcal{A}, τ) , f_1, f_2 are semicircular and c is circular (see [19]). In fact we can get more.

Lemma 7. [16] *Let $(\mathcal{A}, E_{\mathcal{B}}, \mathcal{B})$ be a \mathcal{B} -valued noncommutative probability space, $X, Y \in (\mathcal{A}, E_{\mathcal{B}}, \mathcal{B})$ be η -semicircular, and let $C \in (\mathcal{A}, E_{\mathcal{B}}, \mathcal{B})$ be η -circular. If $X, Y, \{C, C^*\}$ is a \mathcal{B} -free family, then $\begin{pmatrix} X & C \\ C^* & Y \end{pmatrix} \in (M_2(\mathcal{A}), E_{\mathcal{B}} \otimes I_2, M_2(\mathcal{B}))$ is a semicircular variable with covariance η^+ , where $\eta^+ \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} := \begin{pmatrix} \eta(b_{11} + b_{22}) & 0 \\ 0 & \eta(b_{11} + b_{22}) \end{pmatrix}$*

Theorem 8. *Let $c \in (\mathcal{A}, E_{\mathcal{B}}, \mathcal{B})$ be E -circular, where $E : \mathcal{B} \rightarrow \mathcal{D}$ is a conditional expectation. Then*

$$(4) \quad \Phi_{\mathcal{B}}^*(c, c^* : \mathcal{B}, E) = 2$$

Proof. Let $\{f_i\} \subseteq \mathcal{B}$ be a normalized tight frame in the Hilbert C^* -module $\mathcal{L}_{\mathcal{D}}^2(\mathcal{B})$ which induced by E . Let $s_1, s_2 \in (\mathcal{A}, E_{\mathcal{B}}, \mathcal{B})$ be two E -semicircular variables such that $s_1, s_2, \{c, c^*\}$ is a free family with amalgamation over \mathcal{B} , and let $\{a, x, y, b\}$ be the conjugate system of $\{s_1, c, c^*, s_2\}$ with respect to E . Then we claim $E_{\mathcal{B}}(ax) = E_{\mathcal{B}}(ay) = E_{\mathcal{B}}(xb) = E_{\mathcal{B}}(yb) = 0$ since $a, \{x, y\}, b$ is a free family. By Theorem 5,

$$(5) \quad \begin{aligned} & \Phi_{M_2(\mathcal{B})}^* \left(\begin{pmatrix} s_1 & c \\ c^* & s_2 \end{pmatrix} : M_2(\mathcal{B}), E^+ \right) \\ &= \begin{pmatrix} \Phi_{\mathcal{B}}^*(s_1; \mathcal{B}, E) + E_{\mathcal{B}}(yy^*) & 0 \\ 0 & \Phi_{\mathcal{B}}^*(s_2; \mathcal{B}, E) + E_{\mathcal{B}}(xx^*) \end{pmatrix} \end{aligned}$$

and from [10], we know $\Phi_{\mathcal{B}}^*(s_1; \mathcal{B}, E) = \Phi_{\mathcal{B}}^*(s_2; \mathcal{B}, E) = 1$.

On the other hand, $\begin{pmatrix} s_1 & c \\ c^* & s_2 \end{pmatrix} \in (M_2(\mathcal{A}), E_{\mathcal{B}} \otimes I_2, M_2(\mathcal{B}))$ is E^+ semicircular.

Note that E^+ is not a conditional expectation, since $E^+ \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \neq \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}$ in general. But for $\left\{ \begin{pmatrix} f_i & 0 \\ 0 & 0 \end{pmatrix} \right\}_i \cup \left\{ \begin{pmatrix} 0 & f_i \\ 0 & 0 \end{pmatrix} \right\}_i \cup \left\{ \begin{pmatrix} 0 & 0 \\ f_i & 0 \end{pmatrix} \right\}_i \cup \left\{ \begin{pmatrix} 0 & 0 \\ 0 & f_i \end{pmatrix} \right\}_i$, we claim that $Y := \sum_i \begin{pmatrix} f_i & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} s_1 & c \\ c^* & s_2 \end{pmatrix} E^+ \left(\begin{pmatrix} f_i^* & 0 \\ 0 & 0 \end{pmatrix} \right) + \sum_i \begin{pmatrix} 0 & 0 \\ 0 & f_i \end{pmatrix} \begin{pmatrix} s_1 & c \\ c^* & s_2 \end{pmatrix} E^+ \left(\begin{pmatrix} 0 & 0 \\ 0 & f_i^* \end{pmatrix} \right)$ is the conjugate variable of $\begin{pmatrix} s_1 & c \\ c^* & s_2 \end{pmatrix}$

with respect to $M_2(\mathcal{B}), E^+$ in $(M_2(\mathcal{A}), E_{\mathcal{B}} \otimes I_2, M_2(\mathcal{B}))$. In fact,

$$\begin{aligned}
& K^{(2)} \left(Y \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} s_1 & c \\ c^* & s_2 \end{pmatrix} \right) \\
&= \sum_i \begin{pmatrix} f_i & 0 \\ 0 & 0 \end{pmatrix} E^+ \begin{pmatrix} f_i^* & 0 \\ 0 & 0 \end{pmatrix} E^+ \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\
&\quad + \sum_i \begin{pmatrix} 0 & f_i \\ 0 & 0 \end{pmatrix} E^+ \begin{pmatrix} 0 & 0 \\ f_i^* & 0 \end{pmatrix} E^+ \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\
&\quad + \sum_i \begin{pmatrix} 0 & 0 \\ f_i & 0 \end{pmatrix} E^+ \begin{pmatrix} 0 & f_i^* \\ 0 & 0 \end{pmatrix} E^+ \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\
&\quad + \sum_i \begin{pmatrix} 0 & 0 \\ 0 & f_i \end{pmatrix} E^+ \begin{pmatrix} 0 & 0 \\ 0 & f_i^* \end{pmatrix} E^+ \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\
&= \begin{pmatrix} E(b_{11} + b_{22}) & 0 \\ 0 & E(b_{11} + b_{22}) \end{pmatrix} = E^+ \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},
\end{aligned}$$

and

$$K^{(m+1)} \left(Y \otimes B_1 \begin{pmatrix} s_1 & c \\ c^* & s_2 \end{pmatrix} \otimes \cdots \otimes B_m \begin{pmatrix} s_1 & c \\ c^* & s_2 \end{pmatrix} \right) = 0,$$

for all $B_1, \dots, B_m \in M_2(\mathcal{B})$, $m \neq 1$, since $\begin{pmatrix} s_1 & c \\ c^* & c^* \end{pmatrix}$ is semicircular.

So,

$$\begin{aligned}
 & \Phi_{M_2(\mathcal{B})}^* \left(\begin{pmatrix} s_1 & c \\ c^* & s_2 \end{pmatrix} : M_2(\mathcal{B}), E^+ \right) \\
 = & (E_{\mathcal{B}} \otimes I_2)(YY^*) \\
 = & (E_{\mathcal{B}} \otimes I_2) \left[\sum_{i,j} \begin{pmatrix} f_i & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} s_1 & c \\ c^* & s_2 \end{pmatrix} E^+ \begin{pmatrix} f_i^* & 0 \\ 0 & 0 \end{pmatrix} E^+ \begin{pmatrix} f_j & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} s_1 & c \\ c^* & s_2 \end{pmatrix} \begin{pmatrix} f_j^* & 0 \\ 0 & 0 \end{pmatrix} \right. \\
 & \sum_{i,j} \begin{pmatrix} 0 & f_i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} s_1 & c \\ c^* & s_2 \end{pmatrix} E^+ \begin{pmatrix} 0 & 0 \\ f_i^* & 0 \end{pmatrix} E^+ \begin{pmatrix} 0 & f_i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} s_1 & c \\ c^* & s_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ f_i^* & 0 \end{pmatrix} \\
 & + \sum_{i,j} \begin{pmatrix} 0 & 0 \\ f_i & 0 \end{pmatrix} \begin{pmatrix} s_1 & c \\ c^* & s_2 \end{pmatrix} E^+ \begin{pmatrix} 0 & f_i^* \\ 0 & 0 \end{pmatrix} E^+ \begin{pmatrix} 0 & 0 \\ f_i & 0 \end{pmatrix} \begin{pmatrix} s_1 & c \\ c^* & s_2 \end{pmatrix} \begin{pmatrix} 0 & f_i^* \\ 0 & 0 \end{pmatrix} \\
 & \left. + \sum_{i,j} \begin{pmatrix} 0 & 0 \\ 0 & f_i \end{pmatrix} \begin{pmatrix} s_1 & c \\ c^* & s_2 \end{pmatrix} E^+ \begin{pmatrix} 0 & 0 \\ 0 & f_i^* \end{pmatrix} E^+ \begin{pmatrix} 0 & 0 \\ 0 & f_i \end{pmatrix} \begin{pmatrix} s_1 & c \\ c^* & s_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & f_i^* \end{pmatrix} \right] \\
 = & (E_{\mathcal{B}} \otimes I_2) \left[\sum_{i,j} \begin{pmatrix} f_i & 0 \\ 0 & 0 \end{pmatrix} E^+ \left(E^+ \begin{pmatrix} f_i^* & 0 \\ 0 & 0 \end{pmatrix} E^+ \begin{pmatrix} f_j & 0 \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} f_j^* & 0 \\ 0 & 0 \end{pmatrix} \right. \\
 & \sum_{i,j} \begin{pmatrix} 0 & f_i \\ 0 & 0 \end{pmatrix} E^+ \left(E^+ \begin{pmatrix} 0 & 0 \\ f_i^* & 0 \end{pmatrix} E^+ \begin{pmatrix} 0 & f_i \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} 0 & 0 \\ f_i^* & 0 \end{pmatrix} \\
 & + \sum_{i,j} \begin{pmatrix} 0 & 0 \\ f_i & 0 \end{pmatrix} E^+ \left(E^+ \begin{pmatrix} 0 & f_i^* \\ 0 & 0 \end{pmatrix} E^+ \begin{pmatrix} 0 & 0 \\ f_i & 0 \end{pmatrix} \right) \begin{pmatrix} 0 & f_i^* \\ 0 & 0 \end{pmatrix} \\
 & + \sum_{i,j} \begin{pmatrix} 0 & 0 \\ 0 & f_i \end{pmatrix} E^+ \left(E^+ \begin{pmatrix} 0 & 0 \\ 0 & f_i^* \end{pmatrix} \right) \begin{pmatrix} 0 & f_i^* \\ 0 & 0 \end{pmatrix} \\
 & \left. + \sum_{i,j} \begin{pmatrix} 0 & 0 \\ 0 & f_i \end{pmatrix} E^+ \left(E^+ \begin{pmatrix} 0 & 0 \\ 0 & f_i^* \end{pmatrix} E^+ \begin{pmatrix} 0 & 0 \\ 0 & f_i \end{pmatrix} \right) \begin{pmatrix} 0 & 0 \\ 0 & f_i^* \end{pmatrix} \right] \\
 = & \begin{pmatrix} 2 & \\ & 2 \end{pmatrix}
 \end{aligned}$$

Compared with Eq.(6), we get $E_{\mathcal{B}}(yy^*) = E_{\mathcal{B}}(xx^*) = 1$ and so

$$\Phi_{\mathcal{B}}(c, c^* : \mathcal{B}, E) = E_{\mathcal{B}}(yy^*) + E_{\mathcal{B}}(xx^*) = 2$$

□

Theorem 9. Let $c \in (\mathcal{A}, E_{\mathcal{B}}, \mathcal{B})$ be E -circular, where $E : \mathcal{B} \rightarrow \mathcal{D}$ is a conditional expectation. Then

$$(6) \quad \Phi_{\mathcal{B}}^*(c, c^* : \mathcal{B}, id) = 2 \text{Index}(E)$$

Proof. Let $\{f_i\} \subseteq \mathcal{B}$ be a normalized tight frame in the Hilbert C^* -module $\mathcal{L}_{\mathcal{D}}^2(\mathcal{B})$ which induced by E . Let $s_1, s_2 \in (\mathcal{A}, E_{\mathcal{B}}, \mathcal{B})$ be two E -semicircular variables such that $s_1, s_2, \{c, c^*\}$ is a free family with amalgamation over \mathcal{B} , and let $\{a, x, y, b\}$ be

the conjugate system of $\{s_1, c, c^*, s_2\}$ with respect to E . Then

$$(7) \quad \begin{aligned} & \Phi_{M_2(\mathcal{B})}^* \left(\begin{pmatrix} s_1 & c \\ c^* & s_2 \end{pmatrix} : M_2(\mathcal{B}), id^+ \right) \\ &= \begin{pmatrix} \Phi_{\mathcal{B}}^*(s_1; \mathcal{B}, id) + E_{\mathcal{B}}(yy^*) & 0 \\ 0 & \Phi_{\mathcal{B}}^*(s_1; \mathcal{B}, id) + E_{\mathcal{B}}(xx^*) \end{pmatrix} \end{aligned}$$

Then similar to the proof of the above theorem, we can say that

$$\sum_i \begin{pmatrix} f_i & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} s_1 & c \\ c^* & s_2 \end{pmatrix} \begin{pmatrix} f_i^* & \\ & f_i^* \end{pmatrix} + \sum_i \begin{pmatrix} 0 & 0 \\ 0 & f_i \end{pmatrix} \begin{pmatrix} s_1 & c \\ c^* & s_2 \end{pmatrix} \begin{pmatrix} f_i^* & 0 \\ 0 & f_i^* \end{pmatrix}$$

is the conjugate variable of $\begin{pmatrix} s_1 & c \\ c^* & s_2 \end{pmatrix}$ with respect to $M_2(\mathcal{B}), id^+$. And so

$$\begin{aligned} & \Phi_{M_2(\mathcal{B})}^* \left(\begin{pmatrix} s_1 & c \\ c^* & s_2 \end{pmatrix} : M_2(\mathcal{B}), id^+ \right) = \begin{pmatrix} 2 \sum_i f_i f_i^* & \\ & 2 \sum_i f_i f_i^* \end{pmatrix} \\ &= \begin{pmatrix} 2Index(E) & \\ & 2Index(E) \end{pmatrix} \end{aligned}$$

Then $\Phi_{\mathcal{B}}^*(c, c^* : \mathcal{B}, id) = 2Index(E)$.

□

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